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# A class of weak Hopf algebras related to a Borcherds–Cartan matrix

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**Abstract**
In this paper we define a new kind of quantized enveloping algebra of a generalized Kac–Moody algebra $\mathcal{G}$. We denote this algebra by $wU_q^\tau(\mathcal{G})$. It is a noncommutative and noncocommutative weak Hopf algebra. It has a homomorphic image which is isomorphic to the usual quantum enveloping algebra $U_q(\mathcal{G})$ of $\mathcal{G}$.

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## 1. Introduction

In his study of Monstrous moonshine [3–5], Borcherds introduced a new class of infinite-dimensional Lie algebras called generalized Kac–Moody algebras. These generalized Kac–Moody algebras have a contravariant bilinear form which is almost positive definite. The fixed point algebra of any Kac–Moody algebra under a diagram automorphism is a generalized Kac–Moody algebra. A generalized Kac–Moody algebra can be regarded as a Kac–Moody algebra with imaginary simple roots. More explicitly, a generalized Kac–Moody algebra is determined by a Borcherds–Cartan matrix $A = (a_{ij})_{(i,j)\in I\times I}$, where either $a_{ii} = 2$, or $a_{ii} \leqslant 0$. If $a_{ii} \leqslant 0$, then the index $i$ is called imaginary, and the corresponding simple root $\alpha_i$ is called an imaginary simple root. In this paper, the set $\{i \in I | a_{ii} = 2\}$ is denoted by $I^+$. Set $I^{\mathrm{im}} = I \setminus I^+$. The structure and the representation theory of generalized Kac–Moody algebras are very similar to those of Kac–Moody algebras, and many basic facts about the latter can be extended to the former. For example, the Weyl–Kac formula for an irreducible representation over a Kac–Moody algebra is generalized to the Borcherds–Weyl–Kac character formula for an irreducible representation over a generalized Kac–Moody algebra as follows. We have

$$\mathrm{ch}V(\lambda) = \frac{\sum_{w\in W}\sum_{F\subseteq T, F\perp\lambda}(-1)^{l(w)+|F|}\,\mathrm{e}^{w(\lambda+\rho-s(F))}}{\sum_{w\in W, F\subseteq T}(-1)^{l(w)+|F|}\,\mathrm{e}^{w(\rho-s(F))}},$$

  

where $T$ is the set of all imaginary simple roots, $F$ runs all over finite subsets of $T$ such that any two elements in $F$ are mutually perpendicular. We denote by $s(F)$ the sum of the roots in $F$.

On the other hand, many mathematicians are interested in the generalization of Hopf algebras, whose importance has been recognized in both mathematics and physics. One way to do this is to introduce a kind of weak co-product such that $\Delta(1) \neq 1 \otimes 1$ in [1]. The face algebras [7] and generalized Kac algebras [16] are examples of this class of weak Hopf algebras. Li and Duplij have defined and studied another kind of weak Hopf algebra [12]. A bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called a weak Hopf algebra if there is an anti-automorphism $T$ such that $T * \mathrm{id}_H * T = \mathrm{id}_H$ and $\mathrm{id}_H * T * \mathrm{id}_H = T$, where $\mathrm{id}_H$ is the identity map and $*$ is the convolution product. Hopf algebras, and left or right Hopf [13, 14] algebras are weak Hopf algebras in this sense. In the present paper a weak Hopf algebra always means the weak Hopf algebra in this sense. The weak quantized enveloping algebras of semi-simple Lie algebras are also weak Hopf algebras [15]. Our aim is to give more nontrivial examples of weak Hopf algebras. Thanks to the definition of quantized enveloping algebra $U_q(\mathcal{G})$ associated with a generalized Kac–Moody algebra $\mathcal{G}$ defined in [7], we can also replace the group $G(U_q(\mathcal{G}))$ of group-like elements by some regular monoid as in [15]; we use a new generator $J$ instead of the projector in [14]. Our new generator $J$ satisfies $J^m = J$ for some integer $m \geqslant 3$. In this way, we obtain a subclass of weak Hopf algebra $wU_q^\tau(\mathcal{G})$. This weak Hopf algebra has a homomorphic image, which is isomorphic to a sub-Hopf algebra of the quantized enveloping algebra $U_q(\mathcal{G})$ as Hopf algebras. As in the case of the classic quantum group $U_q(\mathcal{G})$, we try to determine irreducible representation of $wU_q(\mathcal{G})$.

Finally, let us outline the structure of this paper. In section 2, we recall some basic facts related to the quantized enveloping algebra of a generalized Kac–Moody algebra. In section 3, we give the definition of $wU_q^\tau(\mathcal{G})$. We study the weak Hopf structure of $wU_q^\tau(\mathcal{G})$ in section 4. In the final section, we study the irreducible representation of $wU_q^\tau(\mathcal{G})$.

## 2. Notations and preliminaries

In this section, we fix notations and recall fundamental results about generalized Kac–Moody algebras.

Let $I = \{1, \ldots, n\}$ or the set of positive integers, and $A = (a_{ij})_{I\times I}$, a Borcherds–Cartan matrix, i.e., it satisfies

(1) $a_{ii} = 2$ or $a_{ii} \leqslant 0$ for all $i \in I$,
(2) $a_{ij} \leqslant 0$ for all $i \neq j$,
(3) $a_{ij} \in \mathbf{Z}$,
(4) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We say that an index $i$ is real if $a_{ii} = 2$ and imaginary if $a_{ii} \leqslant 0$. We denote $I^+ = \{i \in I | a_{ii} = 2\}$ and $I^{\mathrm{im}} = I - I^+$. In addition, we assume that no $a_{ii} = 0$ and all $a_{ii} \in 2\mathbf{Z}$.

In [9] Kang considered Borcherds–Cartan matrices with charge

$$\mathbf{m} = \{(m_i \in \mathbf{Z}_{\geqslant 0}) | i \in I, m_i = 1 \text{ for } i \in I^+\}.$$

The charge $m_i$ is the multiplicity of the simple root corresponding to $i \in I$. In this paper, we follow [11] and assume that $m_i = 1$ for all $i \in I$. However, we do not lose generality by this hypothesis. Indeed, if we take Borcherds–Cartan matrices with some of the rows and columns identical, then the generalized Kac–Moody algebras with charge introduced in [9] can be recovered from those in the present paper by identifying the $h_i s$ and $d_i s$( and hence the $\alpha_i s$) corresponding to these identical rows and columns. So we always assume $m_i = 1$ for all $i \in I$.

Moreover, we also assume that $A$ is symmetrizable, that is, there is a diagonal matrix $D = \mathrm{diag}\{0 < s_i \in \mathbf{Z} | i \in I\}$ such that $DA$ is a symmetric matrix.

Let $P^\vee = (\oplus_{i\in I}\mathbf{Z}h_i) \oplus (\oplus_{i\in I}\mathbf{Z}d_i)$ be a free Abelian group generated by the set $\{h_i, d_i | i \in I\}$. This free Abelian group is called the co-weight lattice of $A$. The element $h_i$ in $\Pi^\vee = \{h_i | i \in I\}$ is called a simple co-weight. We call $\Pi^\vee$the set of all simple co-weights. The space $\mathcal{H} = \mathbf{Q}\otimes_{\mathbf{Z}} P^\vee$ over the rational number field $\mathbf{Q}$ is said to be a Cartan subalgebra. The weight lattice is defined to be $P := \{\lambda \in \mathcal{H}^* | \lambda(P^\vee) \subseteq \mathbf{Z}\}$, where $\mathcal{H}^*$ is the dual space of the Cartan subalgebra $\mathcal{H} = \mathbf{Q}\otimes_{\mathbf{Z}} P^\vee$. We denote by $P^+$ the set $\{\lambda \in P | \lambda(h_i) \geqslant 0$, for every $i \in I\}$ of dominant integral weights.

Define $\alpha_i, \Lambda_i \in \mathcal{H}^*$ by

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d_j) = \delta_{ij}$$
$$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d_j) = 0.$$

Then $\alpha_i, i \in I$ are called simple roots of $A$. Let $\Pi = \{\alpha_i | i \in I\} \subset P$ be the set of simple roots. The free Abelian group $Q = \oplus_{i\in I}\mathbf{Z}\alpha_i$ is called the root lattice. Set $Q_+ = \sum_{i\in I}\mathbf{Z}_{\geqslant 0}\alpha_i$ and $Q_- = -Q_+$. For any $\alpha \in Q_+$, we can write $\alpha = \sum_{k=1}^{n}\alpha_{i_k}$ for $i_1, i_2, \ldots, i_n \in I$. We set $ht(\alpha) = n$ and call it the height of $\alpha$.

Let $(.|.)$ be the bilinear form on $(\oplus_i(\mathbf{Q}\alpha_i \oplus \mathbf{Q}\Lambda_i)) \times \mathcal{H}^*$ defined by

$$(\alpha_i|\lambda) = s_i\lambda(h_i), \qquad (\Lambda_i|\lambda) = s_i\lambda(d_i).$$

Since it is symmetric on $(\oplus_i(\mathbf{Q}\alpha_i \oplus \mathbf{Q}\Lambda_i)) \times (\oplus_i(\mathbf{Q}\alpha_i \oplus \mathbf{Q}\Lambda_i))$, one can extend this to a symmetric bilinear form on $\mathcal{H}^*$. Then such a form is nondegenerate.

We always assume that $\mathbf{K}$ is a field of characteristic 0. Let $q$ be an indeterminate variable over $\mathbf{K}$ and $q_i = q^{s_i}$. For an indeterminant $\nu$ and an integer $m$, let

$$[m]_\nu = \frac{\nu^m - \nu^{-m}}{\nu - \nu^{-1}}, \qquad [m]!_\nu = [m]_\nu \cdots [1]_\nu, [0]_\nu = 1,$$

and

$$\begin{bmatrix} m \\ s \end{bmatrix}_\nu = \frac{[m]!_\nu}{[s]!_\nu [m-s]!_\nu}.$$

**Definition 2.1.** *The quantum generalized Kac–Moody algebra $U'_q(\mathcal{G})$ associated with a Borcherds–Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra with unit 1 over a field $\mathbf{K}$ of characteristic 0, generated by the symbols $e_i, f_i (i \in I)$ and $P^\vee$ subject to the following defining relations:*

$$q^0 = 1, \qquad q^h q^{h'} = q^{h+h'} \quad \forall h, h' \in P^\vee,$$
$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \qquad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i,$$
$$e_i f_j - f_j e_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \qquad \textit{where} \quad k_i = q^{s_i h_i},$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \qquad \textit{if} \quad a_{ii} = 2, i \neq j,$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r = 0 \qquad \textit{if} \quad a_{ii} = 2, i \neq j,$$
$$e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0, \qquad \textit{if} \quad a_{ij} = 0.$$

The quantum generalized Kac–Moody algebra $U'_q(\mathcal{G})$ has a Hopf algebra structure with the co-multiplication $\Delta$, the co-unit $\varepsilon$, and antipode $S$ defined by

$$\Delta(q^h) = q^h \otimes q^h,$$
$$\Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i,$$
$$\Delta(f_i) = k_i \otimes f_i + f_i \otimes 1,$$
$$\varepsilon(q^h) = 1, \varepsilon(e_i) = \varepsilon(f_i) = 0,$$
$$S(q^h) = q^{-h}, S(e_i) = -e_i k_i, S(f_i) = -k_i^{-1} f_i$$

for all $h \in P^\vee$ and $i \in I$.

Let $U_q^+(\mathcal{G})$ and $U_q^-(\mathcal{G})$ be the subalgebras of $U'_q(\mathcal{G})$ generated by elements $e_i$ and $f_i$ respectively, for $i \in I$, and let $U_q^{0\prime}(\mathcal{G})$ be the subalgebra of $U_q(\mathcal{G})$ generated by $q^h (h \in P^\vee)$. Then we have the triangular decomposition [1, 8]

$$U'_q(\mathcal{G}) = U_q^-(\mathcal{G}) \otimes U_q^{0\prime}(\mathcal{G}) \otimes U_q^+(\mathcal{G}).$$

Denote by $U_q(\mathcal{G})$ the subalgebra of $U'_q(\mathcal{G})$ generated by $e_i, f_i, q^{\pm s_i h_i}, q^{\pm d_i}$. Then the triangular decomposition of $U'_q(\mathcal{G})$ induces a triangular decomposition of $U_q(\mathcal{G})$ as follows:

$$U_q(\mathcal{G}) = U_q^-(\mathcal{G}) \otimes U_q^0(\mathcal{G}) \otimes U_q^+(\mathcal{G}),$$

where $U_q^0(\mathcal{G}) = U_q^{0\prime}(\mathcal{G}) \cap U_q(\mathcal{G})$.

## 3. The $\tau$-type algebras $wU_q^\tau(\mathcal{G})$

Since the characteristic of the field $\mathbf{K}$ is equal to zero, $\frac{1}{m} \in \mathbf{K}$ for any nonzero integer $m$. Let $m$ be a fixed positive integer. To generalize the invertibility condition $k_i k_i^{-1} = 1$ in $U_q(\mathcal{G})$, let us introduce some new generators $J$, $K_i$ and $\bar{K}_i$, which are subject to the following relations:

$$J^m = J, \qquad J = K_i \bar{K}_i = \bar{K}_i K_i = D_i \bar{D}_i = \bar{D}_i D_i. \tag{3.1}$$

Suppose $K_i$ and $\bar{K}_i$ are not zero divisors. Then

$$K_i J^{m-1} = J^{m-1} K_i = K_i, \qquad \bar{K}_i J^{m-1} = J^{m-1} \bar{K}_i = \bar{K}_i. \tag{3.2}$$

$$D_i J^{m-1} = J^{m-1} D_i = D_i, \qquad \bar{D}_i J^{m-1} = J^{m-1} \bar{D}_i = \bar{D}_i. \tag{3.3}$$

Although we get (3.2) and (3.3) by the assumption that $K_i$ and $\overline{K_i}$ are not zero divisors, we do not assume that $K_i$ and $\overline{K_i}$ are not zero divisors. We only assume that (3.2) and (3.3) hold in this paper. We call an element $E_i$ of type $m-1$ if it satisfies

$$K_j E_i = q_i^{a_{ij}} E_i K_j, \qquad \bar{K}_j E_i = q_i^{-a_{ij}} E_i \bar{K}_j. \tag{3.4}$$

Similarly, if

$$K_j F_i = q_i^{-a_{ij}} F_i K_j, \qquad \bar{K}_j F_i = q_i^{a_{ij}} F_i \bar{K}_j, \tag{3.5}$$

then $F_i$ is said to be type $m-1$. Suppose

$$K_j E_i J^t \bar{K}_j = q_i^{a_{ij}} E_i J^{t+1}, \qquad E_i J = J E_i, \qquad E_i J^{m-1} = E_i \tag{3.6}$$

for some $0 \leqslant t \leqslant m-2$, then we say that $E_i$ is type $t$.

Similarly, $F_i$ is type $t$ if it satisfies the following:

$$K_j F_i J^t \bar{K}_j = q_i^{-a_{ij}} F_i J^{t+1}, \qquad F_i J = J F_i, \qquad F_i J^{m-1} = F_i. \tag{3.7}$$

From the above definitions, one can obtain the following result.

**Proposition 3.1.** *(1) $E_i$ is type $t$ for $0 \leqslant t \leqslant m-2$ if and only if $E_i J^{t+1}$ is type $m-1$ and $E_i J = J E_i$.*

*(2) $F_i$ is type $t$ for $0 \leqslant t \leqslant m-2$ if and only if $F_i J^{t+1}$ is type $m-1$ and $F_i J = J F_i$.*

**Proof.** The proof of (1) is similar to that of (2). So we only give the proof of (1).

If $E_i$ is type $t$, then

$$K_j E_i J^{t+1} = K_j E_i J^t J = K_j E_i J^t \bar{K}_j K_j = q_i^{a_{ij}} E_i K_j J^{t+1},$$

and

$$\bar{K}_j \big( q_i^{a_{ij}} E_i J^{t+1} \big) = \bar{K}_j (K_j E_i J^t \bar{K}_j) = J E_i \bar{K}_j J^t = E_i J^{t+1} \bar{K}_j.$$

So $E_i J^{t+1}$ is type $m-1$ by definition. Conversely, if $E_i J^{t+1}$ is type $m-1$, then

$$K_j E_i J^t \bar{K}_j = K_j E_i J^t J^{m-1} \bar{K}_j = q_i^{a_{ij}} E_i J^{t+1} K_j J^{m-2} K_j = q_i^{a_{ij}} E_i J^{t+1},$$

and

$$E_i J = E_i K_j \bar{K}_j = q_i^{-a_{ij}} K_j E_i \bar{K}_j = K_j \bar{K}_j E_i = J E_i.$$

That is, $E_i$ is type $t$ satisfying $J E_i = E_i J$. This completes the proof. □

The types of $E_i$ and $F_i$ are denoted by $\kappa_i, \bar{\kappa}$, respectively. Let $\tau = (\{\kappa_i\}_{i \in I} | \{\bar{\kappa}_i\}_{i \in I})$. The $\tau$ is called admissible if it satisfies the following condition:

(1) If $\kappa_i = t$, then $\bar{\kappa}_i = t$ for $1 \leqslant t \leqslant m-2$.
(2) If $\kappa_i = 0$, then $\bar{\kappa}_i = 0, m-1$.
(3) If $\kappa_i = m-1$, then $\bar{\kappa}_i = 0, m-1$.

Now, we can give the definition of the weak quantum algebra of type $\tau$ as follows.

**Definition 3.1.** *Suppose $\tau$ is admissible. The type $\tau$ weak quantum algebra $wU_q^\tau(\mathcal{G})$ associated the generalized Kac–Moody algebra $\mathcal{G}$ an associative algebra with unit 1 over a field $\mathbf{K}$ of characteristic 0, generated by the symbols $J$, which is in the centre of this algebra, $E_i, F_i (i \in I)$ and $K_i, D_i (i \in I)$ subject to the following defining relations:*

$$J^m = J = K_i \bar{K}_i = D_i \bar{D}_i, \tag{3.8}$$

$$K_i \bar{K}_j = \bar{K}_j K_i, \qquad K_i K_j = K_j K_i, \qquad \bar{K}_i \bar{K}_j = \bar{K}_j \bar{K}_i, \tag{3.9}$$

$$D_i \bar{D}_j = \bar{D}_j D_i, \qquad D_i D_j = D_j D_i, \qquad \bar{D}_i \bar{D}_j = \bar{D}_j \bar{D}_i, \tag{3.10}$$

$$D_i \bar{K}_j = \bar{K}_j D_i, \qquad K_i D_j = D_j K_i, \qquad \bar{D}_i K_j = K_j \bar{D}_i, \tag{3.11}$$

$$E_i \quad F_i \quad \text{are type} \quad \tau, \tag{3.12}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - \bar{K}_i}{q_i - q_i^{-1}}, \tag{3.13}$$

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_i E_i^{1-a_{ij}-r}E_jE_i^r=0 \qquad \textit{if} \quad a_{ii}=2, i\neq j, \tag{3.14}$$

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_i F_i^{1-a_{ij}-r}F_jF_i^r=0 \qquad \textit{if} \quad a_{ii}=2, i\neq j, \tag{3.15}$$

$$E_iE_j-E_jE_i=F_iF_j-F_jF_i=0, \qquad \textit{if} \quad a_{ij}=0. \tag{3.16}$$

If $m=1$, then we set $J^0=1$. Since $P^\vee$ is spanned by $h_i, d_i$, $wU_q^T(\mathcal{G})=U_q(\mathcal{G})$ provided that we identify $K_i$ with $q^{s_ih_i}$, $\bar{K}_i$ with $q^{-s_ih_i}$, $D_i$ with $q^{d_i}$ and $\bar{D}_i$ with $q^{-d_i}$. If $m=2$ and $\mathcal{G}$ is a semi-simple Lie algebra, then $wU_q^T(\mathcal{G})$ has been defined and studied by Yang in [5]. Note that the type zero was called type two by Yang. The following conclusion can be proved directly from definition.

**Proposition 3.2.** *(1) $E_i$ is type $t$ for $0\leqslant t\leqslant m-2$ if and only if $E_i$ is type $m-1$ and $E_iJ^{m-1}=J^{m-1}E_i=E_i$.*

*(2) $F_i$ is type $t$ for $0\leqslant t\leqslant m-2$ if and only if $F_i$ is type $m-1$ and $F_iJ^{m-1}=J^{m-1}F_i$.*

**Proof.** The proof of (1) is similar to that of (2). So we only give the proof of (1).

If $E_i$ is type $t$ for $0\leqslant t\leqslant m-2$, then $K_jE_iJ^{t+1}=q_i^{a_{ij}}E_iJ^{t+1}K_j$ by proposition 3.1. Hence,

$$K_jE_i=K_jE_iJ^{m-1}=K_jE_iJ^{t+1}J^{m-t-2}=q_i^{a_{ij}}E_iJ^{t+1}K_jJ^{m-t-2}=q_i^{a_{ij}}E_iK_j,$$

and

$$\bar{K}_jE_i=\bar{K}_jE_iJ^{m-1}=\bar{K}_jE_iJ^{t+1}J^{m-t-2}=q_i^{-a_{ij}}E_iJ^{t+1}\bar{K}_jJ^{m-t-2}=q_i^{-a_{ij}}E_i\bar{K}_j.$$

This proves the claim that $E_i$ is type $m-1$.

Conversely, if $E_i$ is type $m-1$ and $J^{m-1}E_i=E_i$, then

$$K_jE_iJ^t\bar{K}_j=q_i^{a_{ij}}E_iJ^t\bar{K}_jK_j=q_i^{a_{ij}}E_iJ^{t+1}.$$

This completes the proof. □

From this proposition, we know that types $t$ for $0\leqslant t\leqslant m-2$ are the same. However, in the next section, we will see that the different types have different co-multiplications.

## 4. The weak Hopf algebra structure of $wU_q^\tau(\mathcal{G})$

Since the (weak) Hopf algebra structure of $wU_q^\tau(\mathcal{G})$ has been studied in the cases $m=1,2$, we always assume that $m\geqslant 3$ in the following. To make the $\tau$-type algebra $wU_q^\tau(\mathcal{G})$ become a weak Hopf algebra, we define three maps,

$$\Delta: wU_q^\tau(\mathcal{G})\to wU_q^\tau(\mathcal{G})\otimes wU_q^\tau(\mathcal{G}),$$
$$\varepsilon: wU_q^\tau(\mathcal{G})\to \mathbf{K},$$
$$T: wU_q^\tau(\mathcal{G})\to wU_q^\tau(\mathcal{G}),$$

as follows:

$$\Delta(K_i)=K_i\otimes K_i, \qquad \Delta(\bar{K}_i)=\bar{K}_i\otimes\bar{K}_i, \tag{4.1}$$

$$\Delta(D_i)=D_i\otimes D_i, \qquad \Delta(\bar{D}_i)=\bar{D}_i\otimes\bar{D}_i, \tag{4.2}$$

$$\Delta(J)=J\otimes J \tag{4.3}$$

$$\Delta(E_i) = J^{m-1-t} \otimes E_i + E_i \otimes K_i J^t, \qquad E_i \text{ is type } t. \tag{4.4}$$

If $t = 0$, then $\Delta(E_i) = J^{m-1} \otimes E_i + E_i \otimes K_i$. If $t = m - 1$, then $\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i$, since $K_i J^{m-1} = K_i$.

$$\Delta(F_i) = F_i \otimes J^{m-1-t} + \bar{K}_i J^t \otimes F_i, \qquad F_i \text{ is type } t. \tag{4.5}$$

$$\varepsilon(K_i) = \varepsilon(\bar{K}_i) = 1, \qquad \varepsilon(D_i) = \varepsilon(\bar{D}_i) = 1, \qquad \varepsilon(J) = 1, \tag{4.6}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \tag{4.7}$$

$$T(1) = 1, \qquad T(K_i) = \bar{K}_i J^{m-2}, \qquad T(\bar{K}_i) = K_i J^{m-2}, \tag{4.8}$$

$$T(J) = J^{m-2}, \qquad T(D_i) = \bar{D}_i J^{m-2}, \qquad T(\bar{D}_i) = D_i J^{m-2}, \tag{4.9}$$

$$T(E_i) = -E_i \bar{K}_i J^{m-2}, \qquad T(F_i) = -K_i F_i J^{m-2}. \tag{4.10}$$

Let us use $\mu$, $\eta$ to denote the multiplication and unit of the algebra $wU_q^\tau(\mathcal{G})$, respectively. In this section, we will prove the following theorem.

**Theorem 4.1.** *$\big(wU_q^\tau(\mathcal{G}), \mu, \eta, \Delta, \varepsilon\big)$ is a weak Hopf algebra.*

We split the proof of this theorem into two lemmas.

**Lemma 4.2.** *$wU_q^\tau(\mathcal{G})$ is a bialgebra with co-multiplication $\Delta$ and co-unit $\varepsilon$.*

**Proof.** It can be shown by direct calculation that the following relations hold:

$$\begin{aligned}
&\Delta(K_i)\Delta(\bar{K}_j) = \Delta(\bar{K}_j)\Delta(K_i), && \Delta(D_i)\Delta(\bar{D}_j) = \Delta(\bar{D}_j)\Delta(D_i),\\
&\Delta(J) = \Delta(K_i)\Delta(\bar{K}_i) = \Delta(D_i)\Delta(\bar{D}_i), &&\\
&\Delta(J^{m-1}K_i) = \Delta(K_i), && \Delta(J^{m-1}\bar{K}_i) = \Delta(\bar{K}_i),\\
&\Delta(J^{m-1}D_i) = \Delta(D_i), && \Delta(J^{m-1}\bar{D}_i) = \Delta(\bar{D}_i),\\
&\varepsilon(K_i\bar{K}_j) = \varepsilon(K_i)\varepsilon(\bar{K}_j), && \varepsilon(D_i\bar{D}_j) = \varepsilon(D_i)\varepsilon(\bar{D}_j),\\
&\varepsilon(J^{m-1}K_i) = \varepsilon(K_i), && \varepsilon(J^{m-1}\bar{K}_j) = \varepsilon(\bar{K}_j),\\
&\varepsilon(J^{m-1}\bar{D}_i) = \varepsilon(D_i), && \varepsilon(J^{m-1}\bar{D}_j) = \varepsilon(\bar{D}_j),\\
&\varepsilon(K_j)\varepsilon(E_i) = q_i^{a_{ij}}\varepsilon(E_i)\varepsilon(K_j), && \varepsilon(F_i)\varepsilon(\bar{K}_j) = q_i^{a_{ij}}\varepsilon(F_j)\varepsilon(\bar{K}_j),\\
&\varepsilon(J^{t+1})\varepsilon(E_i) = \varepsilon(E_i), && \varepsilon(F_i)\varepsilon(J^{t+1}) = \varepsilon(F_j),\\
&\varepsilon(E_i)\varepsilon(F_j) - \varepsilon(F_j)\varepsilon(E_i) = \delta_{ij}\frac{\varepsilon(K_i) - \varepsilon(\bar{K}_i)}{q_i - q_i^{-1}}.
\end{aligned}$$

If $E_i$ is type $m - 1$, then

$$\begin{aligned}
\Delta(K_j)\Delta(E_i) &= (K_j \otimes K_j)((1 \otimes E_i + E_i \otimes K_i)\\
&= K_j \otimes K_j E_i + K_j E_i \otimes K_j K_i\\
&= q_i^{a_{ij}}\Delta(E_i)\Delta(K_j).
\end{aligned}$$

If $E_i$ is type $t$ for $0 \leqslant t \leqslant m - 2$, then

$$\begin{aligned}
\Delta(K_j)\Delta(E_i)\Delta(J^{t+1}) &= (K_j \otimes K_j)((J^{m-1-t} \otimes E_i + E_i \otimes K_i J^t)(J^{t+1} \otimes J^{t+1})\\
&= K_j J^m \bar{K}_j \otimes K_j E_i J^{t+1} + K_j E_i J^{t+1} \otimes K_j K_i J^{t+1}\\
&= q_i^{a_{ij}}\Delta(E_i)\Delta(J^{t+1})\Delta(K_j),
\end{aligned}$$

and

$$\Delta(E_i)\Delta(J) = (J^{m-t} \otimes E_i J + E_i J \otimes K_i J^{t+1}) = \Delta(J)\Delta(E_i).$$

Hence, $\Delta(K_j)\Delta(E_i)\Delta(J^t)\Delta(\bar{K}_j) = q_i^{a_{ij}}\Delta(E_i)\Delta(J^{t+1})$ for $0 \leqslant t \leqslant m-2$. Similarly, one can prove

$$\Delta(K_j)\Delta(F_i)\Delta(J^t)\Delta(\bar{K}_j) = q_i^{-a_{ij}}\Delta(F_i)\Delta(J^{t+1})$$

for $0 \leqslant t \leqslant m-2$ provided that $F_i$ is type $t$, and

$$\Delta(K_j)\Delta(F_i) = q_i^{-a_{ij}}\Delta(F_i)\Delta(K_j), \qquad \Delta(\bar{K}_j)\Delta(F_i) = q_i^{a_{ij}}\Delta(F_i)\Delta(\bar{K}_j)$$

if $F_i$ is type $m-1$. Next we prove that

$$\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) = \delta_{ij}\frac{\Delta(K_i) - \Delta(\bar{K}_i)}{q_i - q_i^{-1}}. \tag{4.11}$$

Suppose $E_i$ and $F_j$ are types $r, s$, respectively, for $0 \leqslant r, s \leqslant m-2$, we have

$$\begin{aligned}
\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) &= (J^{m-1-r} \otimes E_i + E_i \otimes K_i J^r)(F_j \otimes J^{m-1-s} + \bar{K}_j J^s \otimes F_j)\\
&\quad - (F_j \otimes J^{m-1-s} + \bar{K}_j J^s \otimes F_j)(J^{m-1-r} \otimes E_i + E_i \otimes K_i J^r)\\
&= J^{m-1-r}F_j \otimes E_i J^{m-1-s} + \bar{K}_j J^{m-r-1+s} \otimes E_i F_j + E_i F_j \otimes K_i J^{m-1-s+r}\\
&\quad + E_i \bar{K}_j J^s \otimes K_i F_j J^r - F_j J^{m-1-r} \otimes J^{m-1-s} E_i - F_j E_i \otimes K_i J^{m-1+r-s}\\
&\quad - \bar{K}_j J^{m-1-r+s} \otimes F_j E_i - \bar{K}_j J^s E_i \otimes F_j K_i J^r\\
&= \bar{K}_j J^{m-1-r+s} \otimes (E_i F_j - F_j E_i) + (E_i F_j - F_j E_i) \otimes K_i J^{m-1-s+r}\\
&= \delta_{ij}\frac{\Delta(K_i) - \Delta(\bar{K}_i)}{q_i - q_i^{-1}}.
\end{aligned}$$

Suppose $E_i$ is type $m-1$ and $F_j$ is type $s$ for $0 \leqslant s \leqslant m-2$, we have

$$\begin{aligned}
\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) &= (1 \otimes E_i + E_i \otimes K_i)(F_j \otimes J^{m-1-s} + \bar{K}_j J^s \otimes F_j)\\
&\quad - (F_j \otimes J^{m-1-s} + \bar{K}_j J^s \otimes F_j)(1 \otimes E_i + E_i \otimes K_i)\\
&= F_j \otimes E_i J^{m-1-s} + \bar{K}_j J^s \otimes E_i F_j + E_i F_j \otimes K_i J^{m-1-s}\\
&\quad + E_i \bar{K}_j J^s \otimes K_i F_j - F_j \otimes J^{m-1-s} E_i - F_j E_i \otimes K_i J^{m-1-s}\\
&\quad - \bar{K}_j J^s \otimes F_j E_i - \bar{K}_j J^s E_i \otimes F_j K_i\\
&= \bar{K}_j J^s \otimes (E_i F_j - F_j E_i) + (E_i F_j - F_j E_i) \otimes K_i J^{m-1-s}\\
&= \delta_{ij}\frac{\Delta(K_i) - \Delta(\bar{K}_i)}{q_i - q_i^{-1}}.
\end{aligned}$$

Similarly, we can prove that

$$\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) = \delta_{ij}\frac{\Delta(K_i) - \Delta(\bar{K}_i)}{q_i - q_i^{-1}},$$

if $E_i$ is type $r$ for $0 \leqslant r \leqslant m-2$ and $F_j$ is type $m-1$. So (4.11) holds for all $i, j$.

Finally, we prove that $\Delta$ satisfies the quantum Serre relations, i.e., the relations from (3.14) to (3.16). For (3.16). If $a_{ij} = 0$, $E_i$ is type $r$ and $E_j$ is type $s$ for $0 \leqslant r, s \leqslant m-2$, then

$$\begin{aligned}
\Delta(E_i)\Delta(E_j) - \Delta(E_j)\Delta(E_i) &= (J^{m-1-r} \otimes E_i + E_i \otimes K_i J^r)(J^{m-1-s} \otimes E_j + E_j \otimes K_j J^s)\\
&\quad - (J^{m-1-s} \otimes E_j + E_j \otimes K_j J^s)(J^{m-1-r} \otimes E_i + E_i \otimes K_i J^r)\\
&= J^{2(m-1)-r-s} \otimes E_i E_j + J^{m-1-r} E_j \otimes E_i K_j J^s + E_i J^{m-1-s} \otimes K_i J^r E_j\\
&\quad + E_i E_j \otimes K_i K_j J^{r+s} - J^{2(m-1)-r-s} \otimes E_j E_i - J^{m-1-s} E_i \otimes E_j K_i J^r\\
&\quad - E_j J^{m-1-r} \otimes K_j J^s E_i - E_j E_i \otimes K_j K_i J^{r+s}\\
&= 0.
\end{aligned}$$

If $a_{ij} = 0$, $E_i$ is type $m-1$ and $E_j$ is type $s$ for $0 \leqslant s \leqslant m-2$, then

$$\begin{aligned}\Delta(E_i)\Delta(E_j) - \Delta(E_j)\Delta(E_i) &= (1 \otimes E_i + E_i \otimes K_i)(J^{m-1-s} \otimes E_j + E_j \otimes K_j J^s)\\ &\quad - (J^{m-1-s} \otimes E_j + E_j \otimes K_j J^s)(1 \otimes E_i + E_i \otimes K_i)\\ &= J^{(m-1)-s} \otimes E_i E_j + E_j \otimes E_i K_j J^s + E_i J^{m-1-s} \otimes K_i E_j\\ &\quad + E_i E_j \otimes K_i K_j J^s - J^{(m-1)-s} \otimes E_j E_i - J^{m-1-s} E_i \otimes E_j K_i\\ &\quad - E_j \otimes K_j J^s E_i - E_j E_i \otimes K_j K_i J^s\\ &= 0.\end{aligned}$$

So $\Delta$ satisfies the relation $E_i E_j - E_j E_i = 0$. Similarly, we can prove $\Delta$ satisfies relation $F_i F_j - F_j F_i = 0$. Thus $\Delta$ satisfies relation (3.16).

Next, we prove that $\Delta$ satisfies relation (3.14). The following four cases need to be considered:

(a) $E_i$ is type $t$, $E_j$ is type $s$, $a_{ii} = 2$, where $0 \leqslant r, s, \leqslant m-2$.

(b) $E_i$ is type $m-1$, $E_j$ is type $s$, $a_{ii} = 2$, where $0 \leqslant s, \leqslant m-2$.

(c) $E_i$ is type $t$, $E_j$ is type $m-1$, $a_{ii} = 2$, where $0 \leqslant r, s, \leqslant m-2$.

(d) Both $E_i$ and $E_j$ are type $m-1$.

The case (d) has been proven in [8, pp 18–19]. The proofs of the other cases adapt from the method of [8, pp 18–19]. So we only give the proof of the case (a). Let $r = 1 - a_{ij}$ and

$$u_{ij} = \sum_{\alpha=0}^{r} (-1)^\alpha \begin{bmatrix} r \\ \alpha \end{bmatrix}_i E_i^{r-\alpha} E_j E_i^\alpha.$$

Since $\Delta(E_i) = J^{m-1-t} \otimes E_i + E_i \otimes K_i J^t$,

$$\begin{aligned}\Delta\big(E_i^a\big) &= J^{(m-1-t)a} \otimes E_i^a + E_i^a \otimes K_i J^{at}\\ &\quad + \sum_{\beta=1}^{a-1} q_i^{\beta(a-\beta)} \begin{bmatrix} a \\ \beta \end{bmatrix}_i J^{(m-1-t)\beta} E_i^{a-\beta} \otimes E_i^\beta K_i^{a-\beta} J^{t(a-\beta)}.\end{aligned}$$

This implies easily that

$$\begin{aligned}\Delta(u_{ij}) &= J^{(m-1-t)r+(m-1-s)} \otimes u_{ij} + u_{ij} \otimes J^{tr+s} K_i^r K_j + \sum_{\xi=1}^{r} J^{(m-1-t)(r-\xi)+(m-1-s)} E_i^\xi \otimes X_\xi\\ &\quad + \sum_{l,n} E_i^l E_j E_i^n J^{(m-1-t)(r-l-n)} \otimes Y_{l,n}\end{aligned}$$

with suitable $X_\xi$ and $Y_{l,n}$, and where the last sum is over the integers $l, n \geqslant 0$ with $l + n < r$. We have to show that all $X_\xi$ and $Y_{l,n}$ are equal to zero.

For all $l, n$ as above the term $Y_{l,n}$ is equal to

$$\begin{aligned}&\sum_{\zeta=n}^{r-l} (-1)^\zeta \begin{bmatrix} r \\ \zeta \end{bmatrix}_i q_i^{l(r-\zeta-l)} \begin{bmatrix} r-\zeta \\ l \end{bmatrix}_i E_i^{r-\zeta-l} K_i^l K_j q_i^{n(\zeta-n)} \begin{bmatrix} \zeta \\ n \end{bmatrix}_i E_i^{\zeta-n} K_i J^{t(l-n)+s}\\ &\quad = \left( \sum_{\zeta=n}^{r-l} (-1)^\zeta \begin{bmatrix} r \\ \zeta \end{bmatrix}_i q_i^{l(r-\zeta-l)} \begin{bmatrix} r-\zeta \\ l \end{bmatrix}_i E_i^{r-\zeta-l} K_i^l \right.\\ &\qquad \left. \times K_j q_i^{n(\zeta-n)} \begin{bmatrix} \zeta \\ n \end{bmatrix}_i E_i^{\zeta-n} K_i \right) J^{t(l-n)+s} = 0.\end{aligned}$$

The term $X_\xi$ ( with $1 \leqslant \xi \leqslant r$) is equal to

$$\sum_{\zeta=0}^{r}(-1)^\zeta \begin{bmatrix} r \\ \zeta \end{bmatrix}_i \sum_\beta q_i^{\beta(r-\zeta-\alpha)} \begin{bmatrix} r-\zeta \\ \beta \end{bmatrix}_i E_i^{r-\zeta-\beta} K_i^\beta E_j$$

$$\times q_i^{(\xi-\beta)(\zeta-\xi-\beta)} \begin{bmatrix} \zeta \\ \xi-\beta \end{bmatrix}_i E_i^{\zeta-\xi-\beta} K_i^{\xi-\beta)} J^{t(r-\xi)}$$

$$= \left( \sum_{\zeta=0}^{r}(-1)^\zeta \begin{bmatrix} r \\ \zeta \end{bmatrix}_i \sum_\beta q_i^{\beta(r-\zeta-\alpha)} \begin{bmatrix} r-\zeta \\ \beta \end{bmatrix}_i E_i^{r-\zeta-\beta} K_i^\beta E_j \right.$$

$$\left. \times q_i^{(\xi-\beta)(\zeta-\xi-\beta)} \begin{bmatrix} \zeta \\ \xi-\beta \end{bmatrix}_i E_i^{\zeta-\xi-\beta} K_i^{\xi-\beta} \right) J^{t(r-\xi)} = 0.$$

By now, we have proved that $\Delta$ satisfies relation (3.14). Similarly, we can prove that $\Delta$ satisfies relation (3.15).

It is easy to verify that

$$(\Delta \otimes 1)\Delta(X) = (1 \otimes \Delta)\Delta(X)$$

for $X = J, E_i, F_i, K_i, \bar{K}_i, D_i, \bar{D}_i$. Since $\Delta$ is an algebra homomorphism, $(\Delta \otimes 1)\Delta(X) = (1 \otimes \Delta)\Delta(X)$ for any $X \in wU_q^\tau(\mathcal{G})$. Similarly, we can prove $(1 \otimes \varepsilon)\Delta(X) = (\varepsilon \otimes 1)\Delta(X) = X$ for any $X \in wU_q^\tau(\mathcal{G})$. So $\left(wU_q^\tau(\mathcal{G}), \Delta, \varepsilon, \mu, \eta\right)$ is a bialgebra, where $\mu$ is the multiplication of the algebra and $\eta$ is the unit of the algebra. □

**Lemma 4.3.** *T is a weak antipode of the bialgebra* $wU_q^\tau(\mathcal{G})$.

**Proof.** First we prove that $T$ can be extended to an anti-automorphism of $wU_q^\tau(\mathcal{G})$. It is easy to prove the following relations are true:

$$T(K_i)T(\bar{K}_j) = T(\bar{K}_j)T(K_i), \qquad T(D_i)T(\bar{D}_j) = T(\bar{D}_j)T(D_i),$$
$$T(D_i)T(\bar{K}_j) = T(\bar{K}_j)T(D_i), \qquad T(K_i)T(\bar{D}_j) = T(\bar{D}_j)T(K_i),$$
$$T(\bar{D}_i)T(\bar{K}_j) = T(\bar{K}_j)T(\bar{D}_i), \qquad T(J^{m-1})T(\bar{K}_i) = T(\bar{K}_i),$$
$$T(J^{m-1})T(K_i) = T(K_i), \qquad T(J^{m-1})T(\bar{D}_i) = T(\bar{D}_i),$$
$$T(J^{m-1})T(D_i) = T(D_i).$$

$$T(E_i)T(E_j) = T(E_j)T(E_i), \qquad T(F_i)T(F_j) = T(F_j)T(F_i), \qquad \textit{if} \quad a_{ij} = 0.$$

If $E_i$ is type $m-1$, then

$$\begin{aligned} T(E_i)T(K_j) &= -E_i J^{m-2} K_i \bar{K}_j J^{m-2} \\ &= -q_i^{a_{ij}} \bar{K}_j J^{m-2} E_i K_i J^{m-2} \\ &= q_i^{a_{ij}} T(K_j)T(E_i). \end{aligned}$$

If $E_i$ is type $t$ for $0 \leqslant t \leqslant m-2$, then

$$\begin{aligned} T(\bar{K}_j)T(J^t)T(E_i)T(K_j) &= -K_j J^{m-2} J^{(m-2)t} E_i J^{m-2} K_i \bar{K}_j J^{m-2} \\ &= -q_i^{a_{ij}} E_i J^{(t+2)(m-2)} K_i, \end{aligned}$$

and

$$\begin{aligned} q_i^{a_{ij}} T(J^{t+1})T(E_i) &= -q_i^{a_{ij}} J^{(m-2)(t+1)} E_i K_i J^{m-2} \\ &= -q_i^{a_{ij}} J^{(t+1)} E_i J^{(m-2)(t+2)} K_i. \end{aligned}$$

Consequently,

$$T(\bar{K}_j)T(J^t)T(E_i)T(K_j) = q_i^{a_{ij}} T(J^{t+1})T(E_i).$$

Similarly, we can prove

$$T(F_i)T(K_j) = q_i^{-a_{ij}} T(K_j)T(F_i)$$

if $F_i$ is type $m-1$, and

$$T(\bar{K}_j)T(J^t)T(F_i)T(K_j) = q_i^{-a_{ij}} T(J^{t+1})T(F_i)$$

if $F_i$ is type $t$ for $0 \leqslant t \leqslant m-2$. Moreover,

$$\begin{aligned} T(F_j)T(E_i) - T(E_i)T(F_j) &= J^{2(m-2)}(K_j(F_jE_i)\bar{K}_i - E_i\bar{K}_iK_jF_j) \\ &= J^{2(m-2)}K_j(F_jE_i - F_jE_i)\bar{K}_i \\ &= \delta_{ij}J^{2(m-2)}K_j\frac{\bar{K}_i - K_i}{q_i - q_i^{-1}}\bar{K}_i \\ &= \delta_{ij}\frac{T(K_i) - T(\bar{K}_i)}{q_i - q_i^{-1}}. \end{aligned}$$

Then we can prove that $T$ satisfies the anti-relation for quantum Serre relations. Suppose $a_{ii} = 2$ and $s = 1 - a_{ij}$. Then

$$\begin{aligned} &\sum_{r=0}^{s}(-1)^r \begin{bmatrix} s \\ r \end{bmatrix} T(E_i)^r T(E_j) T(E_i)^{s-r} \\ &\quad = (-1)^{s+1} J^{(s+1)(m-2)} \sum_{r=0}^{s}(-1)^r \begin{bmatrix} s \\ r \end{bmatrix} (E_i\bar{K}_i)^r (E_j\bar{K}_j)(E_i\bar{K}_i)^{s-r} \\ &\quad = (-1)^{s+1} J^{\mu} q_i^{\nu} \sum_{r=0}^{s}(-1)^r \begin{bmatrix} s \\ r \end{bmatrix} q_j^{-a_{ji}r} q_i^{-a_{ij}(s-r)} \bar{K}_i^s E_i^r E_j E_i^{s-r} \bar{K}_j \\ &\quad = (-1)^{s+1} J^{(s+1)(m-2)} q_i^{\frac{1}{2}a_{ii}((s-1)s} q_j^{-a_{ji}s} \bar{K}_i^s \left(\sum_{r=0}^{s}(-1)^r \begin{bmatrix} s \\ r \end{bmatrix} E_i^r E_j E_i^{s-r}\right) \bar{K}_j \\ &\quad = 0, \end{aligned}$$

where $\mu = (s+1)(m-2)$, $\nu = \frac{1}{2}a_{ii}(s-1)s$. Similarly, we can prove that

$$\sum_{r=0}^{s}(-1)^r \begin{bmatrix} s \\ r \end{bmatrix} T(F_i)^r T(F_j) T(F_i)^{s-r} = 0 \qquad \textit{if} \quad a_{ii} = 2.$$

From the above discussion, we get that $T$ is an anti-automorphism of $wU_q^{\tau}(\mathcal{G})$. Finally, we prove that $T * \mathrm{id} * T = T$ and $\mathrm{id} * T * \mathrm{id} = \mathrm{id}$. It is easy to verify $T * \mathrm{id} * T(X) = T(X)$ and $\mathrm{id} * T * \mathrm{id}(X) = X$ for $X = K_i, \bar{K}_i, D_i, \bar{D}_i, J$.

Suppose $E_i$ is type $t$ for $0 \leqslant t \leqslant m-2$. Then

$$(1 \times \Delta)\Delta(E_i) = J^{m-1-t} \otimes J^{m-1-t} \otimes E_i + J^{m-1-t} \otimes E_i \otimes K_iJ^t + E_i \otimes K_iJ^t \otimes K_iJ^t.$$

So

$$\begin{aligned} T * \mathrm{id} * T(E_i) &= -J^{(m-1-t)(m-1)}E_iJ^{(m-2)}\bar{K}_i \\ &\quad + J^{(m-1-t)(m-2)}E_i\bar{K}_iJ^{t(m-2)} - E_i\bar{K}_iJ^{m-2+t}K_i\bar{K}_iJ^{(m-2)t} \\ &= -E_iJ^{(m-2)}\bar{K}_i + E_i\bar{K}_i - E_i\bar{K}_i \\ &= T(E_i). \end{aligned}$$

$$\begin{aligned} \mathrm{id} * T * \mathrm{id}(E_i) &= J^{(m-1-t)(m-1)}E_i - J^{(m-1-t)}E_iJ^{m-2}\bar{K}_iK_iJ^t + E_i\bar{K}_iJ^{(m-2)t+m-2}K_i\bar{K}_iJ^t \\ &= E_i. \end{aligned}$$

Suppose $E_i$ is type $m-1$. Then

$$(1\times\Delta)\Delta(E_i) = 1\otimes 1\otimes E_i + 1\otimes E_i\otimes K_i + E_i\otimes K_i\otimes K_i.$$

Hence,

$$\mathrm{id} * T * \mathrm{id}(E_i) = E_i - E_i J^{m-2}\bar{K}_i K_i + E_i\bar{K}_i J^{m-2} K_i = E_i,$$

and

$$T * \mathrm{id} * T(E_i) = -E_i J^{m-2}\bar{K}_i + E_i J^{m-2}\bar{K}_i - E_i J^{m-2}\bar{K}_i K_i \bar{K}_i J^{m-2} = T(E_i).$$

Similarly we can prove that $\mathrm{id} * T * \mathrm{id}(F_i) = F_i$ and $\mathrm{id} * T * \mathrm{id}(F_i) = F_i$.

Since

$$\mathrm{id} * T * \mathrm{id} = (\mu\otimes 1)\mu(\mathrm{id}\otimes T\otimes\mathrm{id})(\Delta\otimes 1)\Delta,$$

$\mathrm{id} * T * \mathrm{id}$ is a linear automorphism of $wU_q^\tau(\mathcal{G})$. To prove $(\mathrm{id} * T * \mathrm{id})(X) = X$, for any $X\in wU_q^\tau(\mathcal{G})$, we only need prove that

$$(\mathrm{id} * T * \mathrm{id})(xy) = xy \tag{4.12}$$

provided that $(\mathrm{id} * T * \mathrm{id})(x) = x$, and $y$ is one of the generators $K_i, \bar{K}_i, D_i, \bar{D}_i, E_i, F_i, J$. Suppose $(\Delta\otimes 1)\Delta(x) = \sum x_{(1)}\otimes x_{(2)}\otimes x_{(3)}$. Then $(\Delta\otimes 1)\Delta(xJ) = \sum x_{(1)}J\otimes x_{(2)}J\otimes x_{(3)}J$ and hence

$$\mathrm{id} * T * \mathrm{id}(xJ) = \sum x_{(1)}JJ^{m-2}T(x_{(2)})x_{(3)}J = xJ.$$

Suppose $E_i$ is type $t$ for $0\leqslant t\leqslant m-2$. Then

$$\begin{aligned}\mathrm{id} * T * \mathrm{id}(xE_i) &= \sum x_{(1)}J^{m-1-t}J^{(m-1-t)(m-2)}T(x_{(2)})x_{(3)}E_i\\ &\quad - \sum x_{(1)}J^{m-1-t}E_i J^{m-2}\bar{K}_i T(x_{(2)})x_{(3)}K_i J^t\\ &\quad + \sum x_{(1)}E_i\bar{K}_i J^{(m-2)t}T(x_{(2)})x_{(3)}K_i J^t\\ &= xE_i,\end{aligned}$$

If $E_i$ is type $m-1$, then

$$\begin{aligned}\mathrm{id} * T * \mathrm{id}(xE_i) &= \sum x_{(1)}T(x_{(2)})x_{(3)}E_i - \sum x_{(1)}E_i J^{m-2}\bar{K}_i T(x_{(2)})x_{(3)}K_i\\ &\quad + \sum x_{(1)}E_i\bar{K}_i J^{(m-2)}T(x_{(2)})x_{(3)}K_i\\ &= xE_i.\end{aligned}$$

We can prove (4.12) is true for other generators of $wU_q^\tau(\mathcal{G})$. So $\mathrm{id} * T * \mathrm{id}(x) = x$ for any $x\in U^\tau(\mathcal{G})$ by induction.

Similarly, we can prove $T * \mathrm{id} * T(x) = T(x)$ for any $x\in wU_q^\tau(\mathcal{G})$. So $T$ is a weak antipode of $wU_q^\tau(\mathcal{G})$, and $wU_q^\tau(\mathcal{G})$ is a weak Hopf algebra. □

**Corollary 4.1.** *$wU^\tau(\mathcal{G})$ is a noncommutative and noncocommutative weak Hopf algebra with the weak antipode $T$, but not a Hopf algebra.*

**Proof.** We only need to prove that it is not a Hopf algebra. If it is a Hopf algebra with an antipode $S$, then $S(J)J = 1$ and $J$ is invertible. This is impossible because $J(J^{m-1}-1) = 0$. □

**Corollary 4.2.** *$wU^\tau(\mathcal{G})/(1-J)$ is isomorphic to the usual quantized enveloping algebra $U_q(\mathcal{G})$.*

Since $J^{m-1}$ is a central idempotent element, $wU_q^\tau(\mathcal{G}) = wU_q^\tau(\mathcal{G})J^{m-1} \oplus wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ as algebras. It is easy to prove that $wU_q^\tau(\mathcal{G})J^{m-1}$ is a subcoalgebra of $wU_q^\tau(\mathcal{G})$. For any co-algebra $H$, the set of group-like elements of $H$ is denoted by $G(H)$. With this notation, we have the following.

**Proposition 4.1.** *(1) $wU_q^\tau(\mathcal{G})J^{m-1} = wU_q^\tau(\mathcal{G})J$ is a sub-weak-Hopf-algebra of $wU_q^\tau(\mathcal{G})$.*

*(2) $G\big(wU_q^\tau(\mathcal{G})\big) = G\big(wU_q^\tau(\mathcal{G})J^{m-1}\big) \cup \{1\}$.*

**Proof.** Since $T(J^{m-1}) = J^{(m-1)(m-2)} = J^{m-1}$, $wU_q^\tau(\mathcal{G})J^{m-1}$ is a sub-weak-Hopf-algebra of $wU_q^\tau(\mathcal{G})$. Moreover, since $J \in wU_q^\tau(\mathcal{G})J^{m-1}$,

$$JwU_q^\tau(\mathcal{G})J^{m-1} = wU_q^\tau(\mathcal{G})J \subseteq wU_q^\tau(\mathcal{G})J^{m-1} \subseteq wU_q^\tau(\mathcal{G})J.$$

Hence, $wU_q^\tau(\mathcal{G})J^{m-1} = wU_q^\tau(\mathcal{G})J$.

By now we have completed the proof of (1). Next, we prove (2).

If $g \in wG\big(U_q^\tau(\mathcal{G})\big)$, then $g = gJ^{m-1} + g(1-J^{m-1})$. Let $g_1 = gJ^{m-1}$, $g_2 = g(1-J^{m-1})$. Then $g \otimes g = \Delta(g) = g_1 \otimes g_1 + g_1 \otimes g_2 + g_2 \otimes g_1 + g_2 \otimes g_2$. Since $\Delta(g_1) = g_1 \otimes g_1$ is a group-like element, $\Delta(g_2) = g_1 \otimes g_2 + g_2 \otimes g_1 + g_2 \otimes g_2$. So

$$\begin{aligned}(1 \otimes \Delta)\Delta(g_2) = {} & g_1 \otimes g_1 \otimes g_2 + g_1 \otimes g_2 \otimes g_1 + g_1 \otimes g_2 \otimes g_2 + g_2 \otimes g_1 \otimes g_1 \\ & + g_2 \otimes g_1 \otimes g_2 + g_2 \otimes g_2 \otimes g_1 + g_2 \otimes g_2 \otimes g_2.\end{aligned}$$

Then

$$\begin{cases}(T * \mathrm{id} * T)(g_2) = T(g_2)g_2T(g_2) = T(g_2), \\ (\mathrm{id} * T * \mathrm{id})(g_2) = g_2T(g_2)g_2 = g_2.\end{cases} \tag{4.13}$$

Because $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is generated by $E_i(1 - J^{m-1})$, $F_j(1 - J^{m-1})$, $1 - J^{m-1}$ and $T(E_i(1 - J^{m-1})) = T(F_j(1 - J^{m-1})) = 0$, $T(g_2) = k(1 - J^{m-1})$ for some $k \in \mathbf{K}$. From (4.13), we obtain the following:

$$\begin{cases}k^2g_2 = k^2(1 - J^m)^2g_2 = k(1 - J^m), \\ kg_2^2(1 - J^m) = kg_2^2 = g_2.\end{cases} \tag{4.14}$$

If $k = 0$, then $g = g_1 \in G\big(wU_q^\tau(\mathcal{G})J^{m-1}\big)$. If $k \neq 0$, then $g_2 = \frac{1}{k}(1 - J^m)$. Thus

$$\frac{1}{k}(1 \otimes 1 - J^m \otimes J^m) = \frac{1}{k}g_1 \otimes (1 - J^m) + \frac{1}{k}(1 - J^m) \otimes g_1 + \frac{1}{k^2}(1 - J^m) \otimes (1 - J^m).$$

Multiplying by $k(J^m \otimes 1)$ on the both sides of the above equation, we get

$$J^m \otimes 1 - J^m \otimes J^m = g_1 \otimes (1 - J^m).$$

Similarly, we have

$$1 \otimes J^m - J^m \otimes J^m = (1 - J^m) \otimes g_1.$$

Then

$$1 \otimes 1 - J^m \otimes J^m = J^m \otimes 1 + 1 \otimes J^m - 2J^m \otimes J^m + \frac{1}{k}(1 - J^m) \otimes (1 - J^m).$$

Hence,

$$(1 - J^m) \otimes (1 - J^m) = \frac{1}{k}(1 - J^m) \otimes (1 - J^m).$$

Consequently, $k = 1$. Note that the set of group-like elements are linearly independent. So we get $g_1 = J^m$ from $1 \otimes J^m - J^m \otimes J^m = (1 - J^m) \otimes J^m = (1 - J^m) \otimes g_1$. Hence, $g = 1$. $\square$

**Proposition 4.2.** *Let* $\bar{J} = \frac{1}{m-1}\sum_{r=1}^{m-1} J^r$. *Then* $\bar{J}$ *is a central idempotent element. Moreover* $wU^\tau(\mathcal{G})\bar{J}$ *is isomorphic to the usual quantized enveloping algebra* $U_q(\mathcal{G})$ *as algebras.*

**Proof.** Define $\phi(E_i\bar{J}) = e_i, \phi(F_i\bar{J}) = f_i, \phi(K_i\bar{J}) = k_i, \phi(D_i\bar{J}) = q_i^{d_i}, \phi(\bar{K}_i\bar{J}) = k_i^{-1}$, $\phi(\bar{D}_i\bar{J}) = q_i^{-d_i}$. It is easy to prove that $\phi$ is a well-defined homomorphism of algebras with inverse mapping $\varphi$ defined as follows. $\varphi(e_i) = E_i\bar{J}, \varphi(f_i) = F_i\bar{J}, \varphi(k_i) = K_i\bar{J}, \varphi(q_i^{d_i}) = D_i\bar{J}, \varphi(1) = \bar{J}$. □

## 5. The representation of $wU_q^\tau(\mathcal{G})$

In this section, we try to determine the irreducible representation of $wU_q^\tau(\mathcal{G})$. Suppose $V$ is a simple module over $wU_q^\tau(\mathcal{G})$. Let $\bar{J} = \frac{1}{m-1}\sum_{r=1}^{m-1} J^r$. Then $\bar{J}$ is a central idempotent element of $wU_q^\tau(\mathcal{G})$ and $\bar{J} = J^{m-1}\bar{J}$. So

$$wU_q^\tau(\mathcal{G}) = wU_q^\tau(\mathcal{G})\bar{J} \oplus wU_q^\tau(\mathcal{G})(J^{m-1} - \bar{J}) \oplus wU_q^\tau(\mathcal{G})(1 - J^{m-1}),$$

is a direct sum of algebras. Let $w_1 = wU_q^\tau(\mathcal{G})\bar{J}$, $w_2 = U_q^\tau(\mathcal{G})(J^{m-1} - \bar{J})$ and $w_3 = wU_q^\tau(\mathcal{G})(1 - J^{m-1})$. Then any module over $wU_q^\tau(\mathcal{G})$ can be decomposed as follows:

$$V = V_1 \oplus V_2 \oplus V_3,$$

where $V_1 = \bar{J}V$, $V_2 = (J^{m-1} - \bar{J})V$ and $V_3 = (1 - J^{m-1})V$. It is obvious that $V_i$ are modules over $w_i$, respectively. If $V$ is a simple $wU_q^\tau(\mathcal{G})$ module, then either $V = V_1$, or $V = V_2$ or $V = V_3$.

If $V = V_1$, then $Jv = v$ for any $v \in V$. Suppose $v \in V$ satisfying $K_iv = \lambda_iv$, then $\lambda_i \neq 0$. In this case, $K_i\bar{K}_iK_iv = \lambda_i^2\bar{K}_i = \lambda_iv$. So $\bar{K}_iv = \frac{1}{\lambda_i}v$.

If $V = V_2$, then $\bar{J}v = 0$ for any $v \in V$. If there is an eigenvector $v$ of $J$, then $Jv = \nu^tv$, where $\nu$ is an $(m-1)$th primitive root of 1 and $t$ is an integer satisfying $1 \leqslant t \leqslant m-1$. Suppose $K_iv = \lambda_iv$, then $\lambda_i \neq 0$. In this case, $K_i\bar{K}_iK_iv = \lambda_i^2\bar{K}_i = \nu^t\lambda_iv$. So $\bar{K}_iv = \frac{\nu^t}{\lambda_i}v$.

If $V = V_3$, then $Jv = 0$ for any $v \in V$, $K_iv = K_iJ^{m-1}v = 0$ and $\bar{K}_iv = \bar{K}_iJ^{m-1}v = 0$ for any $v \in V$.

By now we have completed the proof of the following proposition.

**Proposition 5.1.** *Let* $V$ *be a simple* $wU_q^\tau(\mathcal{G})$ *module. Then either* $Jv = v$ *for all* $v \in V$, *or* $Jv = 0$ *for any* $v \in V$, *or* $\bar{J}v = 0$ *for any* $v \in V$. *Suppose there exists an* $i \in I$ *such that* $K_iv = \lambda_iv$ *for some nonzero vector* $v$. *Then* $\bar{K}_iv = \bar{\lambda}_iv$ *for some* $\lambda_i$, *where*

$$\bar{\lambda}_i = \begin{cases} \lambda_i^{-1}, & \text{if } \lambda_i \neq 0;\ Jv = v \\ \frac{\nu^t}{\lambda_i^{-1}}, & \text{if } \lambda_i \neq 0;\ \bar{J}v = 0,\ v \text{ is an eigenvector of } J \\ 0, & \text{if } \lambda_i = 0, \end{cases}$$

*and* $\nu$ *is a primitive root of 1 and* $t$ *is an integer satisfying* $1 \leqslant t \leqslant m-1$. *Moreover,* $\lambda_i \neq 0$ *if and only if* $Jv \neq 0$.

Suppose $V = V_1$. Then $V$ can be viewed as a module over $wU_q^\tau(\mathcal{G})/(1 - J)$. Note that $wU_q^\tau(\mathcal{G})/(1 - J)$ is isomorphic to $U_q(\mathcal{G})$ as algebras. In this case, $V$ has been studied by Kang [9]. For example, the limit $q \to 1$ of highest weight simple module is a highest weight simple module over the generalized Kac–Moody algebra $\mathcal{G}$ with the same weight $\lambda$. Then this simple module is uniquely determined by its formal Borcherds–Kac–Weyl character formula (see section 1).

Suppose $V = V_3$. Then $JV = 0$ and $K_i V = \bar{K}_i V = 0$ for any $i \in I$ by proposition 5.1. Similarly, we can prove that $D_i V = \bar{D}_i V = 0$ for all $i \in I$. Hence, $E_i F_j V = F_j E_i V$ for all $i, j \in I$. Moreover, $V$ can be viewed as a module over $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$. Recall that $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is generated by $E_i(1 - J^{m-1})$, and $F_j(1 - J^{m-1})$, where $E_i, F_j$ are type $m - 1$. Hence, $E_i(1 - J^{m-1})F_j(1 - J^{m-1})V = F_j(1 - J^{m-1})E_i(1 - J^{m-1})V$ for all $i, j \in I$. In the following, we try to determine the structure of $V$ in some special cases. Let $X_i = E_i(1 - J^{m-1}), Y_i = F_i(1 - J^{m-1})$. Then every simple module $V$ over $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is a module over the algebra generated by $\{X_i, Y_j |$ for $i \in I_1, j \in I_2\}$, where $I_1 = \{i \in I | E_i$ is type $m - 1\}$, $I_2 = \{j \in I | F_j$ is type $m - 1\}$. The generators $X_i, Y_j$ satisfy the following relation:

$$
\begin{aligned}
&X_i Y_j = Y_j X_i, \\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i X_i^{1-a_{ij}-r} X_j X_i^r = 0 \qquad \text{if} \quad a_{ii} = 2, i \neq j, \\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i Y_i^{1-a_{ij}-r} Y_j Y_i^r = 0 \qquad \text{if} \quad a_{ii} = 2, i \neq j, \\
&X_i X_j - X_j X_i = Y_i Y_j - Y_j Y_i = 0, \qquad \text{if} \quad a_{ij} = 0.
\end{aligned}
$$

This simple module $V$ satisfies $JV = 0$. From the above discussion, we obtain the following result.

**Corollary 5.1.** *If $a_{ij} = 0$ for any $j \neq i$, where $i \in I_1 \cap I^+$, $j \in I_2$, then every simple module over $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is isomorphic to*

$$wU_q^\tau(\mathcal{G})(1 - J^{m-1})/M,$$

*where $M$ is a maximal ideal of $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$.*

In the case that $I_1 \cup I_2$ is a finite set, we can obtain the following results. By corollary 5.1, the only simple over $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is $\mathbf{K}[x]/(p(x))$ if $|I_1 \cup I_2| = 1$, where $p(x)$ is an irreducible polynomial in $\mathbf{K}[x]$. Suppose $\mathbf{K}$ is an algebraically closed field. If $a_{ij} = 0$ for any $i \in (I_1 \cup I_2) \cap I^+$, and $|I_1 \cup I_2| = n$, then the simple module $V$ over $wU_q^\tau(\mathcal{G})(1 - J^{m-1})$ is isomorphic to $\mathbf{K}[X_i, Y_j | i \in I_1, j \in I_2]/(\{X_i - a_i, Y_j - b_j | i \in I_1, j \in I_2\})$ for some $\big((a_i)_{i \in I_1}, (b_j)_{j \in I_2}\big) \in \mathbf{K}^n$.


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